Discrete Mathematics 41 (1982) 155–160
North-Holland Publishing Company

ON DUMONT'S POLYNOMIALS

Mourad E.H. ISMAIL and Donald STEWART

Department of Mathematics, Arizona State University, Tempe, AZ 85281, USA

Received 1 May 1981

We derive two generating functions and an explicit formula for the polynomials $\{H_n(x)\}$ studied by Dumont.

Introduction

A Seidel matrix $\{a_{k,n}\}$ is an infinite matrix of rational entries such that

$$a_{k,n} = a_{k-1,n} + a_{k-1,n+1}. \tag{1.1}$$

The first row $(k=0)$ is called the initial sequence and the first column $(n=0)$ is the terminal sequence. It is known that if $a(t)$ is the exponential generating function of the initial sequence

$$\sum_0^\infty a_{0,n}\frac{t^n}{n!} = a(t),$$

then the exponential generating function of the Seidel matrix is

$$\sum_{k,n=0}^\infty a_{k,n}\frac{t^n u^k}{n!\,k!} = e^u a(t+u). \tag{1.2}$$

The generating function (1.2) implies that the only symmetric or skew symmetric Seidel matrix is the zero matrix. It is then natural to seek Seidel matrices with maximal symmetry or skew symmetry. Two such matrices are uniquely determined by

$$b_{0,0}=1, \quad b_{1,0}=-b_{0,1} \quad \text{and} \quad b_{k,0}=b_{0,k} \quad \text{for } k>1, \tag{1.3}$$

$$g_{0,0}=0, \quad g_{1,0}=g_{0,1}=1 \quad \text{and} \quad g_{k,0}=-g_{0,k} \quad \text{for } k>1, \tag{1.4}$$

respectively. the initial sequence of $\{b_{k,n}\}$ is

$$b_{0,2n}=(-1)^{n+1}B_{2n}, \quad b_{0,0}=1, \quad b_{0,1}=-\tfrac{1}{2}, \quad b_{0,2n+1}=0, \quad n\geq 1, \tag{1.5}$$

where $\{B_n\}$ are the Bernoulli numbers. The initial sequence of $\{g_{k,n}\}$ is related to the Genocchi numbers $\{G_n\}$ via

$$g_{0,0}=0, \quad g_{0,1}=1, \quad g_{0,2n+1}=0, \quad g_{0,2n}=(-1)^n G_n \quad \text{for } n>0. \tag{1.6}$$

The first few entries are shown in Table 1.

* Research partially supported by NSF contract MCS80-02539.

Table 1

0	1	−1	0	1	0	−3	0	17	0	−155
1	0	−1	1	1	−3	−3	17	17	−155	
1	−1	0	2	−2	−6	14	34	−138		
0	−1	2	0	−8	8	48	−104			
−1	1	2	−8	0	56	−56				
0	3	−6	−8	56	0					
3	−3	−14	48	56						
0	−17	34	104							
−17	17	138								
0	155									
155										

Gandhi [4] introduced the polynomials $\{Q_n(t)\}$

$$Q_0(t)=1, \qquad Q_{n+1}(t)=t^2Q_n(t+1)-(t-1)^2Q_n(t) \tag{1.7}$$

and conjectured that $Q_n(1)$ is G_{2n-2} or equivalently

$$\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}Q_{n-k-1}(1)=0, \quad n=1,2,\ldots. \tag{1.8}$$

Gandhi's conjecture was proved by Carlitz [1] and independently by Riordan and Stein [5]. Dumont [2] gave a combinatorial interpretation of Gandhi's conjecture.

The entries in the superdiagonal of the matrix in Table 1 also have a combinatorial interpretation, see Viennot and Dumont [6]. Motivated by Gandhi's conjecture, Dumont [3] raised the question of finding generating functions and explicit formulas for the polynomials $\{H_n(x)\}$ defined by

$$H_1(x)=x, \qquad H_{n+1}(x)=x(x+1)\{H_n(x+1)-H_n(x)\}, \quad n>1. \tag{1.9}$$

Dumont [3] also conjectured the following analogue of Gandhi's conjecture.

Dumont's Conjecture. The numbers H'_n defined via

$$H'_0=1, \qquad H'_n=H_n(1) \quad \text{for } n>0, \tag{1.10}$$

satisfy

$$\sum_{k=0}^{n}\binom{n+k}{2k}(-1)^kH'_{n-k}=0, \quad n>0. \tag{1.11}$$

In the present note we establish the following generating functions

$$\sum_1^\infty H_n(x)t^{n-1}=\sum_1^\infty (x)_nt^{n-1}(n-1)!\prod_1^{n-1}\{1+tj(j+1)\}^{-1} \tag{1.12}$$

and

$$\sum_1^\infty \frac{H_n(x)}{n!}t^n=\sum_1^\infty p_n(t)(x)_n \tag{1.13}$$

where

$$(x)_0 := 1, \qquad (x)_n := x(x+1)\cdots(x+n-1), \tag{1.14}$$

and

$$p_n(t) = \frac{1}{n!}\left(t - \frac{n-1}{n}\right) + \sum_{k=2}^{n} \frac{(-1)^k(2k-1)(n-1)!}{k(k-1)(n-k)!\,(n+k-1)!} e^{k(1-k)t} \tag{1.15}$$

The generating functions (1.12) and (1.13) will be proved in Section 2. The power series in (1.12) converges for $x<0$ and $t \neq -1/j(j+1)$, $j=1,2,\ldots$. The series terminates when x is zero or a negative integer. One can easily obtain explicit formulas for $H_n(x)$ from the above generating functions. For example (1.13) implies

$$H_m(x) = \sum_{n=2}^{m} (x)_n \sum_{k=2}^{n} \frac{(-1)^{k-1}(2k-1)(n-1)!}{(n-k)!\,(n+k-1)!} k^{m-1}(1-k)^{m-1}, \quad m>1.$$

In Section 3 we use the generating function (1.12) to derive a generating function for the expression

$$f_n(x) := \sum_{0}^{n} \binom{n+k}{2k}(-1)^k H_{n-k}(x) \tag{1.16}$$

with $H_0(x)=1$.

2. Proofs

Proof of (1.12). Denote the right side of (1.12) by $G(x,t)$. It is easy to see that

$$tx(x+1)[G(x+1,t)-G(x,t)] = \sum_{1}^{\infty} \frac{t^n(n-1)!\,(x+1)n(x)_n}{\prod_{1}^{n-1}[1+tj(j+1)]}$$

$$= \sum_{1}^{\infty} \frac{n!\,t^n(x)_n(x+n+1-n)}{\prod_{1}^{n-1}[1+tj(j+1)]}$$

$$= \sum_{1}^{\infty} \frac{t^n n!\,(x)_{n+1}}{\prod_{j=1}^{n-1}[1+tj(j+1)]} + \sum_{1}^{\infty} \frac{t^n n!\,(x)_n(1-n)}{\prod_{1}^{n-1}[1+tj(j+1)]}$$

$$= \sum_{2}^{\infty} \frac{t^{n-1}(n-1)!\,(x)_n[1+tn(n-1)]}{\prod_{1}^{n-1}[1+tj(j+1)]} + \sum_{1}^{\infty} \frac{t^n n!\,(x)_n(1-n)}{\prod_{1}^{n-1}[1+tj(j+1)]}$$

$$= \sum_{2}^{\infty} \frac{t^{n-1}(n-1)!\,(x)_n}{\prod_{1}^{n-1}[1+tj(j+1)]} = G(x,t) - x.$$

This establishes the functional equation

$$tx(x+1)[G(x+1,t)-G(x,t)] = G(x,t) - x,$$

which is equivalent to having the coefficient of t^{n-1} in $G(x, t)$, say $r_n(x)$, generated by

$$r_{n+1}(x) = x(x+1)[r_n(x+1) - r_n(x)], \quad n \geq 1,$$

with $r_1(x) = x$. This identifies $r_n(x)$ as $H_n(x)$. This completes the proof of (1.12).

We now establish a lemma needed to prove (1.13).

Lemma. *the following identity holds*

$$\sum_{k=2}^{n} \frac{(-1)^k(2k-1)}{k(k-1)} \binom{2n-1}{n-k} = \frac{(2n-1)!\,(n-1)}{(n!)^2}. \tag{2.1}$$

Proof. Express $(2k-1)/k(k-1)$ as $1/k + 1/k-1$. Denote the left side of (2.1) by A. Clearly

$$\begin{aligned}
A &= \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \binom{2n-1}{n-k-1} + \sum_{2}^{n} \frac{(-1)^k}{k} \binom{2n-1}{n-k} \\
&= \binom{2n-1}{n-2} + \frac{(-1)^n}{n} + \sum_{2}^{n-1} \frac{(-1)^k}{k} \left\{ \binom{2n-1}{n-k} - \binom{2n-1}{n-k-1} \right\} \\
&= \binom{2n-1}{n-2} + \frac{(-1)^n}{n} + \sum_{2}^{n-1} \frac{(-1)^k}{k} \frac{(2n-1)!}{(n-k)!\,(n+k)!} \{n+k-(n-k)\} \\
&= \binom{2n-1}{n-2} + \frac{(-1)^n}{n} + \sum_{2}^{n-1} \frac{(-1)^k}{n} \binom{2n}{n-k} \\
&= \binom{2n-1}{n-2} + \sum_{2}^{n} \frac{(-1)^k}{n} \binom{2n}{n-k} \\
&= \binom{2n-1}{n-2} + \frac{1}{n} \binom{2n}{n-1} + \sum_{1}^{n} \frac{(-1)^k}{n} \binom{2n}{n-k} \\
&= \binom{2n-1}{n-2} + \frac{1}{n} \binom{2n}{n-1} - \frac{1}{2n} \binom{2n}{n} + \frac{1}{2} \sum_{-n}^{n} \frac{(-1)^k}{n} \binom{2n}{n-k} \\
&= \binom{2n-1}{n-2} + \frac{1}{n} \binom{2n}{n-1} - \frac{1}{2n} \binom{2n}{n} + \frac{1}{2} \sum_{0}^{2n} \frac{(-1)^{n-k}}{n} \binom{2n}{k} \\
&= \binom{2n-1}{n-2} + \frac{1}{n} \binom{2n}{n-1} - \frac{1}{2n} \binom{2n}{n},
\end{aligned}$$

by the binomial theorem. The above sum of binomial coefficients can be easily reduced to the right side of (2.1).

Finally we come to the proof of (1.13).

Proof of (1.13). We set

$$H(x, t) = \sum_{1}^{\infty} H_n(x) t^n / n! \tag{2.2}$$

The recursion (1.9) is equivalent to

$$\frac{\partial H(x,t)}{\partial t}-x=x(x+1)[H(x+1,t)-H(x,t)].$$

We now express $H(x,t)$ in terms of the shifted factorials $(x)_n$. The relationship $H_n(-1)=-\delta_{n,1}$ follows from (1.9). Let

$$H(x,t)=\sum_1^{\infty} p_n(t)(x)_n. \tag{2.3}$$

Using some elementary manipulations we arrive at

$$\sum_1^{\infty} p_n'(t)(x)_n-x=\sum_1^{\infty} np_n(t)(x)_{n+1}-\sum_1^{\infty} n(n-1)p_n(t)(x)_n.$$

Equating the coefficients of $(x)_{n+1}$ we obtain the following system of differential recurrence relations

$$p_{n+1}'(t)+n(n+1)p_{n+1}(t)=np_n(t), \quad n>0, \qquad p_1(t)=t. \tag{2.4}$$

In order to simplify the computations we put

$$q_n(t):=e^{n(n-1)t}p_n(t) \tag{2.5}$$

and rewrite (2.4) as

$$q_{n+1}'(t)=ne^{2nt}q_n(t), \quad n>0, \qquad q_1(t)=t. \tag{2.6}$$

Observe that the power series (2.2) has no constant term, hence the constant term in $q_n(t)$ must vanish. Combining this observation with (2.6) we see that the q's have the form

$$q_n(t)=\frac{e^{n(n-1)t}}{n!}\left(t-\frac{n-1}{n}\right)+\sum_{k=2}^{n}\binom{n-1}{k-1}\frac{(2k-1)!}{(n+k-1)!}c_k e^{(n+k-1)(n-k)t} \tag{2.7}$$

where the c_k's are defined recursively by requiring $q_n(t)$ to vanish at $t=0$. After computing the first few c's we recognized the pattern

$$c_k=\frac{(-1)^k(k-1)!}{(2k-2)!\,k(k-1)}, \quad k>1. \tag{2.8}$$

Actually it is easy to see that the q_n's defined by (2.7) satisfy (2.6). So it only remains to show that the right side of (2.7) vanishes at $t=0$, that is

$$\sum_{k=2}^{n}\binom{n-1}{k-1}\frac{(2k-1)}{(n+k-1)!}\,\frac{(-1)^k(k-1)!}{k(k-1)}=\frac{(n-1)}{n!\,n}.$$

The above identity is essentially (2.1). This completes the proof.

3. A generating function for $F_n(x)$

Set $F(x,t)=\sum_0^\infty f_n(x)t^{2n}$. Clearly

$$F(x,t)=\sum_{n\geq k\geq 0}^{\infty}\binom{n+k}{2k}(-1)^k t^{2n}H_{n-k}(x)$$

$$=\sum_{n=0}^{\infty}H_n(x)t^{2n}\sum_{k=0}^{\infty}\binom{n+2k}{2k}(-1)^k t^{2k}.$$

The above relationship and the observation

$$2\sum_{k=0}^{\infty}\binom{n+2k}{2k}u^{2k}=\sum_0^\infty\binom{n+k}{k}u^k+\sum_0^\infty\binom{n+k}{k}(-u)^k$$

$$=(1-u)^{-n-1}+(1+u)^{-n-1}$$

establish the following theorem.

Theorem. *The f_n's of* (1.16) *have the generating function*

$$\sum_0^\infty f_n(x)t^n=(1+t)^{-1}+\frac{\frac{1}{2}t}{(1-\mathrm{i}\sqrt{t})^2}G\left(x,\frac{t}{1-\mathrm{i}\sqrt{t}}\right)+\frac{\frac{1}{2}t}{(1+\mathrm{i}\sqrt{t})^2}G\left(x,\frac{t}{1+\mathrm{i}\sqrt{t}}\right) \tag{3.1}$$

where $G(x,t)$ denotes the right side of (1.12).

The above proof uses only formal power series but it is easy to see that (3.1) holds for $x\leq 0$ and $t/(1\pm\mathrm{i}\sqrt{t})\neq -1/j(j+1)$.

Acknowledgement

We would like to express our thanks to Dominique Dumont for suggesting this problem, to Gregory Nielson for very instructive discussions, and to Ruth Camuse for her patience and excellent typing.

References

[1] L. Carlitz, A conjecture concerning Genocchi numbers, Kongelige Norske Videnskabers Selskab 9 (1971) 1–4.

[2] D. Dumont, Sur une conjecture de Gandhi concernant les nombres de Genocchi, Discrete Math. 1 (1972) 321–327.

[3] D. Dumont, Private communication.

[4] J.M. Gandhi, A conjectured representation of Genocchi numbers, Amer. Math. Monthly 77 (1970) 505–506.

[5] J. Riordan and P.R. Stein, Proof of a conjecture on Genocchi numbers, Discrete Math. 5 (1973) 381–388.

[6] G. Viennot and D. Dumont, A combinatorial interpretation of the Seidel generation of Genocchi numbers, to appear.